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TEMPERATURE FIELD OF A THREE-LAYER SPHERE

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This paper presents a solution of the problem of unsteady heat transfer in a three-layer hollow sphere in a central-symmetric formulation with various time-dependent boundary conditions on the inner and outer surfaces. In each layer of the sphere there is heat release of known intensity which depends on the radial coordinate and time. The solution is obtained by a finite integral transform on the radial coordinate. A numerical solution is presented for one version of the boundary conditions.

Key words: *unsteady heat transfer, three-layer sphere, integral transformation, boundary conditions.*

Heat-transfer processes in three-layer and multilayered plates and in two-layer cylinders and spheres with time-independent boundary conditions and constant initial temperature have been considered previously [1, 2]. These problems were solved using the Laplace integral transform. For cylinders and spheres with a number of layers more than two, the heat-transfer problem is difficult to solve using this method. A solution of the heat-conduction problem for an infinite plate with a linear initial condition is presented in [3].

This paper presents a solution of the heat transfer problem in a three-layer hollow sphere with boundary conditions depending on an arbitrary function of time and the initial condition containing an arbitrary function of the radius. In each layer of the sphere there is heat release which depends on time and the radial coordinate. The solution of the problem is obtained using the method of finite integral transform employed in [4, 5] which allows for an easy solution of the problem with different boundary conditions with a minimum change in the initial algorithm.

Below, as an example we give a solution of the heat transfer problem for a three-layer sphere (Fig. 1) with the following boundary conditions: the boundary condition of the second kind are imposed on the inner surface of the first layer, and the condition of the third kind on the outer surface of the third layer; complete thermal contact is assumed on the boundaries between the layers (the boundary condition of the fourth kind).

The problem reduces to solving the equations

$$\frac{1}{\chi_i}\frac{\partial T_i}{\partial \tau} = \frac{\partial^2 T_i}{\partial r^2} + \frac{2}{r}\frac{\partial T_i}{\partial r} + \frac{w_i(\tau, r)}{\lambda_i}, \quad \tau > 0, \qquad T_i = T_i(\tau, r), \quad i = 1, 2, 3,$$
$$a \leqslant r \leqslant b \text{ at } i = 1, \qquad b \leqslant r \leqslant c \text{ at } i = 2, \qquad c \leqslant r \leqslant d \text{ at } i = 3 \tag{1}$$

with the boundary conditions

$$\lambda_1 \frac{\partial T_1}{\partial r}\Big|_{r=a} + q(\tau) = 0, \quad T_1\Big|_{r=b} = T_2\Big|_{r=b}, \quad \lambda_1 \frac{\partial T_1}{\partial r}\Big|_{r=b} = \lambda_2 \frac{\partial T_2}{\partial r}\Big|_{r=b},$$
$$T_2\Big|_{r=c} = T_3\Big|_{r=c}, \qquad \lambda_2 \frac{\partial T_2}{\partial r}\Big|_{r=c} = \lambda_3 \frac{\partial T_3}{\partial r}\Big|_{r=c}, \tag{2}$$
$$\lambda_3 \frac{\partial T_3}{\partial r}\Big|_{r=d} + \alpha\Big[T_3\Big|_{r=d} - T_s(\tau)\Big] = 0$$

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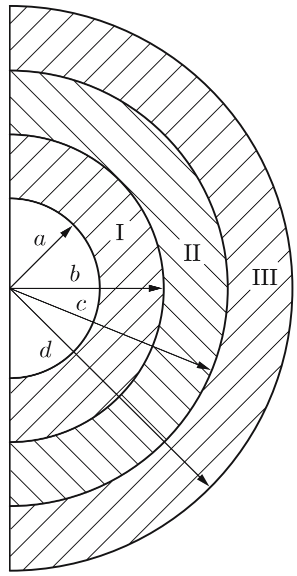

Fig. 1. Geometry of a three-layer sphere: $a \leqslant r \leqslant b$ (first layer), $b \leqslant r \leqslant c$ (second layer), and $c \leqslant r \leqslant d$ (third layer).

and the initial conditions

$$T_i\Big|_{\tau=0} = T_{i0}(r), \qquad i = 1, 2, 3. \tag{3}$$

Here T_i is the temperature in the ith layer of the sphere, λ_i and χ_i are the thermal conductivity and thermal diffusivity, w_i is the heat-release power in the volume of the layer, q is the heat flux on the inner surface of the sphere, α is the heat-transfer coefficient on the outer surface of the sphere, T_s is the temperature of the medium, and a, b, c, and d are the radii of the three-layer sphere.

The change of variables

$$r = \sqrt{\chi_1}\, x \text{ at } a \leqslant r \leqslant b, \quad r = \sqrt{\chi_2}\, x \text{ at } b \leqslant r \leqslant c, \quad r = \sqrt{\chi_3}\, x \text{ at } c \leqslant r \leqslant d \tag{4}$$

reduces relations (1)–(3) to the following form:

$$\frac{\partial V_i}{\partial \tau} = \frac{\partial^2 V_i}{\partial x^2} + \frac{2}{x}\frac{\partial V_i}{\partial x} + \frac{\chi_i}{\lambda_i} w_{ii}, \quad V_i(\tau, x) = T_i(\tau, r), \quad x_{2i-1} \leqslant x \leqslant x_{2i}, \quad i = 1, 2, 3; \tag{5}$$

$$b_1 \frac{\partial V_1}{\partial x}\Big|_{x=x_1} + q(\tau) = 0, \qquad V_1\Big|_{x=x_2} = V_2\Big|_{x=x_3}, \qquad b_1 \frac{\partial V_1}{\partial x}\Big|_{x=x_2} = b_2 \frac{\partial V_2}{\partial x}\Big|_{x=x_3},$$

$$V_2\Big|_{x=x_4} = V_3\Big|_{x=x_5}, \qquad b_2 \frac{\partial V_2}{\partial x}\Big|_{x=x_4} = b_3 \frac{\partial V_3}{\partial x}|_{x=x_5}, \tag{6}$$

$$b_3 \frac{\partial V_3}{\partial r}\Big|_{x=x_6} + \alpha\Big[V_3\Big|_{x=x_6} - T_s(\tau)\Big] = 0;$$

$$V_i\Big|_{\tau=0} = V_{0i}(x), \qquad i = 1, 2, 3. \tag{7}$$

Here

$$w_{ii}(\tau, x) = w_i(\tau, r) \quad (i = 1, 2, 3), \quad x_1 = \frac{a}{\sqrt{\chi_1}}, \quad x_2 = \frac{b}{\sqrt{\chi_1}}, \quad x_3 = \frac{b}{\sqrt{\chi_2}}, \quad x_4 = \frac{c}{\sqrt{\chi_2}},$$

$$x_5 = \frac{c}{\sqrt{\chi_3}}, \quad x_6 = \frac{d}{\sqrt{\chi_3}}, \quad b_1 = \frac{\lambda_1}{\sqrt{\chi_1}}, \quad b_2 = \frac{\lambda_2}{\sqrt{\chi_2}}, \quad b_3 = \frac{\lambda_3}{\sqrt{\chi_3}}.$$

To solve system (5)–(7), we construct a finite integral transform.

Let s and $U_i(sx)$ be the eigenvalues and eigenfunctions of the homogeneous problem corresponding to the initial problem (5), (6):

$$\frac{d^2U_i}{dx^2} + \frac{2}{x}\,\frac{dU_i}{dx} + s^2U_i = 0, \qquad U_i = U_i(sx), \qquad x_{2i-1} \leqslant x \leqslant x_{2i}, \qquad i = 1, 2, 3; \tag{8}$$

$$\frac{\partial U_1}{\partial x}\Big|_{x=x_1} = 0, \qquad U_1\Big|_{x=x_2} = U_2\Big|_{x=x_3}, \qquad b_1\,\frac{dU_1}{dx}\Big|_{x=x_2} = b_2\,\frac{dU_2}{dx}\Big|_{x=x_3},$$

$$U_2\Big|_{x=x_4} = U_3\Big|_{x=x_5}, \qquad b_2\,\frac{dU_2}{dx}\Big|_{x=x_4} = b_3\,\frac{dU_3}{dx}\Big|_{x=x_5}, \tag{9}$$

$$b_3\,\frac{dU_3}{dx}\Big|_{x=x_6} + \alpha U_3\Big|_{x=x_6} = 0.$$

As the kernel of the integral transform of the function $V_i(\tau, x)$ we use the functions $U_i(sx)$:

$$\bar{V}(\tau, s) = A_1\int\limits_{x_1}^{x_2} x^2V_1(\tau, x)\,U_1(sx)\,dx + A_2\int\limits_{x_3}^{x_4} x^2V_2(\tau, x)\,U_2(sx)\,dx + A_3\int\limits_{x_5}^{x_6} x^2V_3(\tau, x)\,U_3(sx)\,dx \tag{10}$$

(x^2 is the weight factor of the spherical coordinate system).

We determine the eigenfunctions $U_i(sx)$ and the eigenvalues s from Eqs. (8) and (9). The solutions of Eqs. (8) have the form

$$U_i(sx) = \frac{1}{x}\,(C_{2i-1}\sin sx + C_{2i}\cos sx), \qquad i = 1, 2, 3, \tag{11}$$

where C_i are undetermined constants. Conditions (9) imply the system of equations

$$\begin{gathered} C_1a_{11} + C_2a_{12} = 0, \qquad C_1a_{21} + C_2a_{22} + C_3a_{23} + C_4a_{24} = 0, \\ C_1a_{31} + C_2a_{32} + C_3a_{33} + C_4a_{34} = 0, \qquad C_3a_{43} + C_4a_{44} + C_5a_{45} + C_6a_{46} = 0, \\ C_3a_{43} + C_4a_{44} + C_5a_{45} + C_6a_{46} = 0, \qquad C_3a_{53} + C_4a_{54} + C_5a_{55} + C_6a_{56} = 0, \\ C_5a_{65} + C_6a_{66} = 0, \end{gathered} \tag{12}$$

where

$$a_{11} = -\frac{\sin sx_1}{x_1^2} + s\,\frac{\cos sx_1}{x_1}, \quad a_{12} = -\frac{\cos sx_1}{x_1^2} - s\,\frac{\sin sx_1}{x_1}, \quad a_{21} = \frac{\sin sx_2}{x_2}, \quad a_{22} = \frac{\cos sx_2}{x_2},$$

$$a_{23} = -\frac{\sin sx_3}{x_3}, \quad a_{24} = -\frac{\cos sx_3}{x_3}, \quad a_{31} = b_1\Big(-\frac{\sin sx_2}{x_2^2} + s\,\frac{\cos sx_2}{x_2}\Big),$$

$$a_{32} = b_1\Big(-\frac{\cos sx_2}{x_2^2} - s\,\frac{\sin sx_2}{x_2}\Big), \quad a_{33} = b_2\Big(\frac{\sin sx_3}{x_3^2} - s\,\frac{\cos sx_3}{x_3}\Big),$$

$$a_{34} = b_2\Big(\frac{\cos sx_3}{x_3^2} + s\,\frac{\sin sx_3}{x_3}\Big), \quad a_{43} = \frac{\sin sx_4}{x_4}, \quad a_{44} = \frac{\cos sx_4}{x_4}, \quad a_{45} = -\frac{\sin sx_5}{x_5},$$

$$a_{46} = -\frac{\cos sx_5}{x_5}, \quad a_{53} = b_2\Big(-\frac{\sin sx_4}{x_4^2} + s\,\frac{\cos sx_4}{x_4}\Big), \quad a_{54} = b_2\Big(-\frac{\cos sx_4}{x_4^2} - s\,\frac{\sin sx_4}{x_4}\Big),$$

$$a_{55} = b_3\Big(\frac{\sin sx_5}{x_5^2} - s\,\frac{\cos sx_5}{x_5}\Big), \quad a_{56} = b_3\Big(\frac{\cos sx_5}{x_5^2} + s\,\frac{\sin sx_5}{x_5}\Big),$$

$$a_{65} = b_3\Big(-\frac{\sin sx_6}{x_6^2} + s\,\frac{\cos sx_6}{x_6}\Big) + \alpha\,\frac{\sin sx_6}{x_6}, \quad a_{66} = b_3\Big(-\frac{\cos sx_6}{x_6^2} - s\,\frac{\sin sx_6}{x_6}\Big) + \alpha\,\frac{\cos sx_6}{x_6}.$$

The solution of system (12) is untrivial only if its determinant vanishes:

$$\begin{vmatrix} a_{11} & a_{12} & 0 & 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & a_{24} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & a_{34} & 0 & 0 \\ 0 & 0 & a_{43} & a_{44} & a_{45} & a_{46} \\ 0 & 0 & a_{53} & a_{54} & a_{55} & a_{56} \\ 0 & 0 & 0 & 0 & a_{65} & a_{66} \end{vmatrix} = 0. \tag{13}$$

The eigennumbers s [2] are determined from Eq. (13).

In system (12), the constants C_2, C_3, C_4, C_5, and C_6 are expressed in terms of C_1. Then, expressions (11) become

$$U_1(sx) = \frac{C_1}{x}(\sin sx + f_1 \cos sx), \qquad U_2(sx) = \frac{C_1}{x}(f_2 \sin sx + f_3 \cos sx),$$
$$U_3(sx) = \frac{C_1}{x}(f_4 \sin sx + f_5 \cos sx). \tag{14}$$

The coefficients f_i $(i = 1, 2, 3, 4, 5)$ contain the quantities a_{11}, a_{12}, a_{21}, $\dots$.

In the integral transform (10) and functions (14), the constants A_1, A_2, A_3, and C_1 remain undetermined. The values of A_1, A_2, and A_3 are determined from the orthogonality condition of the functions $U_i(sx)$:

$$\int_{x_1}^{x_2} x^2 U_1(px)\, U_1(sx)\, dx = 0, \qquad \int_{x_3}^{x_4} x^2 U_2(px)\, U_2(sx)\, dx = 0, \qquad \int_{x_5}^{x_6} x^2 U_3(px)\, U_3(sx)\, dx = 0$$

for $p \neq s$ and from boundary conditions (9). As a result, we obtain

$$A_1 = \frac{x_3^2 b_1}{x_2^2 b_2}, \qquad A_2 = 1, \qquad A_3 = \frac{x_4^2 b_3}{x_5^2 b_2}.$$

The quantity C_1 is determined from the orthonormalization condition for the functions $U_i(sx)$ $(i = 1, 2, 3)$

$$\sum_{i=1}^{3} A_i \int_{x_{2i-1}}^{x_{2i}} x^2 [U_i(sx)]^2\, dx = 1.$$

This implies that

$$C_1 = (J_1 + J_2 + J_3)^{-1/2},$$

where

$$J_1 = A_1\Big(\frac{(1 + f_1)(x_2 - x_1)}{2} + \frac{f_1^2 - 1}{4s} S_1 - \frac{f_1}{2s} P_1\Big),$$

$$J_2 = A_2\Big(\frac{(f_2^2 + f_3^2)(x_4 - x_3)}{2} + \frac{f_3^2 - f_2^2}{4s} S_2 - \frac{f_2 f_3}{2s} P_2\Big),$$

$$J_3 = A_3\Big(\frac{(f_4^2 + f_5^2)(x_6 - x_5)}{2} + \frac{f_5^2 - f_4^2}{4s} S_3 - \frac{f_4 f_5}{2s} P_3\Big),$$

$$S_1 = \sin 2sx_2 - \sin 2sx_1, \qquad P_1 = \cos 2sx_2 - \cos 2sx_1,$$

$$S_2 = \sin 2sx_4 - \sin 2sx_3, \qquad P_2 = \cos 2sx_4 - \cos 2sx_3,$$

$$S_3 = \sin 2sx_6 - \sin 2sx_5, \qquad P_3 = \cos 2sx_6 - \cos 2sx_5.$$

Thus, the integral transform (10) is determined.

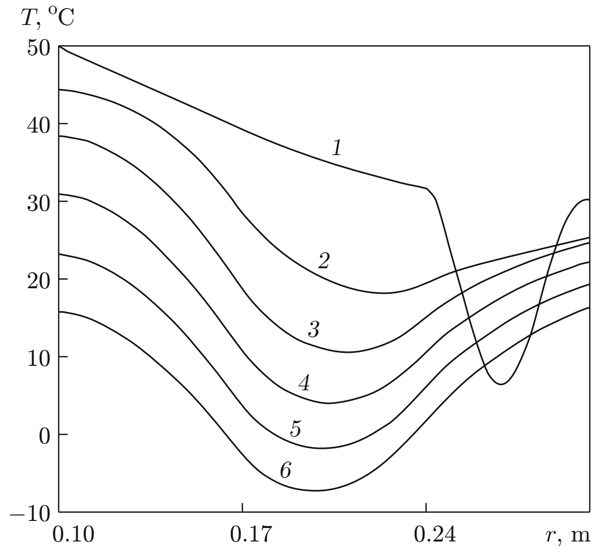

Fig. 2. Temperature distribution along the thickness of the three-layer sphere for $\tau = 0$ (1), 0.25 (2), 0.50 (3), 0.75 (4), 1.0 (5), and 1.25 h (6).

Applying the integral transform to Eq. (5), we obtain

$$\frac{d\bar{V}}{d\tau} + s^2\bar{V} = FP_1(\tau, s) + FP_2(\tau, s), \tag{15}$$

where

$$FP_1(\tau, s) = \frac{A_1 x_1^2}{b_1}\, U_1(sx_1) q(\tau) + \frac{A_3 x_6^2}{b_3}\, U_3(sx_6) T_s(\tau)\alpha,$$
$$FP_2(\tau, s) = \sum_{i=1}^{3} A_i \,\frac{\chi_i}{\lambda_i} \int\limits_{x_{2i-1}}^{x_{2i}} x^2 w_{ii} U_i \, dx. \tag{16}$$

The initial condition for Eq. (15) is obtained from (7) by applying the integral transform (10):

$$\bar{V}_0 = \sum_{i=1}^{3} A_i \int\limits_{x_{2i-1}}^{x_{2i}} x^2 U_i(sx)\, V_{0i}(x)\, dx. \tag{17}$$

The solution of Eq. (15) with the initial condition (17) becomes

$$\bar{V}(\tau, s) = \exp\,(-s^2\tau)\Big(\int\limits_0^{\tau} (FP_1(\tau, s) + FP_2(\tau, s)) \exp\,(s^2\tau)\, d\tau + \bar{V}_0(s)\Big).$$

Because the functions $U_i(sx)$ are orthonormalized, the solution of the problem has the form

$$V_i(\tau, x) = \sum_{s_j} \bar{V}(\tau, s_j)\, U_i(s_j x), \qquad i = 1, 2, 3$$

(summation is performed over the eigenvalues s). Conversion from the coordinates x to the coordinates r is performed using relations (4).

Let us show what changes should be made in the obtained solution in conversion to different boundary conditions on the inner and outer surfaces of the three-layer sphere. We assume the condition of the first kind on the inner surface of the first layer and the condition of the second kind on the outer surface of the third layer. Then, in conditions (2), the first and last boundary conditions should be replaced by the following:

$$T_1\Big|_{r=a} = T_a(\tau), \qquad \lambda_3\, \frac{\partial T_3}{\partial r}\Big|_{r=d} - q(\tau) = 0.$$

In this case, in conditions (9), the first and last boundary conditions are changed accordingly:

$$U_1\Big|_{x=x_1} = 0, \qquad \frac{\partial U_3}{\partial x}\Big|_{x=x_6} = 0.$$

As a result, in Eq. (12), the expressions for a_{11}, a_{12}, a_{65}, and a_{66} become

$$a_{11} = \sin sx_1, \quad a_{12} = \cos sx_1, \quad a_{65} = -\frac{\sin sx_6}{x_6^2} + s\,\frac{\cos sx_6}{x_6}, \quad a_{66} = -\frac{\cos sx_6}{x_6^2} - s\,\frac{\sin sx_6}{x_6}.$$

Expression (16) for $FP_1(\tau, s)$ is written as

$$FP_1(\tau, s) = A_1 x_1^2 \frac{dU_1}{dx}\Big|_{x=x_1} T_a(\tau) + \frac{A_3 x_6^2}{b_3} U_3(sx_6) q(\tau).$$

The remaining expressions remain unchanged.

Numerical Example. Below we give a solution of the initial problem for the following parameter values: $a = 0.1$ m, $b = 0.17$ m, $c = 0.24$ m, $d = 0.3$ m, $q = 2.1 \cdot 10^2$ W/m^2, $w_1 = w_3 = 0$, $w_2 = -2.94$ W/m^3, $\lambda_1 = 0.95$ W/(m$\cdot$ K), $\lambda_2 = 1.24$ W/(m$\cdot$ K), $\lambda_3 = 1.55$ W/(m$\cdot$ K), $a_1 = 0.58 \cdot 10^{-5}$ m^2/sec, $a_2 = 0.75 \cdot 10^{-5}$ m^2/sec, $a_3 = 1.35 \cdot 10^{-5}$ m^2/sec, $T_s = 293$ K, $T_{01}(r) = 1.538\,33r + 338.3833$ [K], $T_{02}(r) = 444/r + 286.0801$ [K], $T_{03}(r) = -334.33 \sin r/r + 291.9891$ [K], and $\alpha = 0.33 \cdot 10^{-2}$ W/(m$^2 \cdot$ K).

Figure 2 shows temperature distributions along the thickness of the three-layer sphere at various times. It is evident that, in 0.25 h, the initial temperature distribution, including the linear, hyperbolic, and sinusoidal hyperbolic regions, changes considerably due to the predominant influence of heat absorption w_2 in the middle layer $b \leqslant r \leqslant c$ and due to the influence of the temperature of the medium T_s on the outer surface of the third layer. On the inner boundary $r = a$, despite the surface heat release q, the temperature also decreases but at a higher rate, which is also a consequence of heat absorption in the middle layer. At the time $\tau = 1.25$ h, the temperatures on the inner surface of the first layer and on the outer surface of the third layer take approximately identical values.

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